

Fermions scattering in a three dimensional extreme black hole background

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The absorption cross section for scattering of fermions off an extreme BTZ black hole is calculated. It is shown that, as in the case of scalar particles, an extreme BTZ black hole exhibits a vanishing absorption cross section, which is consistent with the vanishing entropy of such object. Additionally, we give a general argument to prove that the particle flux near the horizon is zero. Finally we show that the *reciprocal space* introduced previously in [1] gives rise to the same result and, therefore, it could be considered as the space where the scattering process takes place in an AdS spacetime.

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I. INTRODUCTION

The presence of quantum effects in gravity is known since Hawking's discovery that a black hole (BH) can evaporate because of such effects [2]. The knowledge of the full theory of quantum gravity will tell us how this process occurs and whether it is possible that the singularities of gravity can be smoothed out.

The full theory of quantum gravity is unknown nowadays, but there are strong evidence suggesting that it originated, in some limit, from string theory. Such conjecture is supported by results that relate BH properties with string theory effects, namely, the BH entropy and its decay rate. For example, it has been established that the entropy of a five dimensional extreme BH corresponds to the degeneracy of BPS states of a string theory [3]. Also, the decay rate of a BH agrees with the decay rate of thermally excited strings, both being proportional to the absorption cross section [4].

In addition, the classical absorption cross section (or greybody factor) for non extreme (regular) BH in four dimensions, turns out to be proportional to the area of the horizon [5]. In this context, great amount of research has been done; regular black holes corresponding to different metrics have been studied semiclassically, coupling local fields to the fixed black hole background, in dimension four [6] and higher [7].

In three dimensions, the existence of a BH solution of the Einstein equations [8] has opened a window into a powerful laboratory to prove results that, in higher dimensions, are very hard to deal with.

Results for the greybody factor and other quantities obtained from the scattering of scalar, fermions and photon fields off such a BH, have been found in $2 + 1$ dimensions [9], [10], [11]. The calculation of the same quantities for extreme BTZ black holes deserves a careful and separate analysis, because regular and extreme BTZ black holes correspond to different physical objects [12] ; they are associated to different topologies of spacetime: the non-extreme $2 + 1$ -dimensional black hole has the topology of the cylinder, while the extreme case has the topology of an annulus.

The spinless relativistic particle in an extreme BTZ background was discussed in detail in a recent paper [1], where it was shown that the absorption cross section for such black hole is zero (the same result can be deduced from [13]). From the thermodynamic point of view, this result can be interpreted as a signal of zero entropy, in agreement with previous

results found in the literature [12].

In this paper we show that the absorption cross section for massive spin 1/2 particles in a 2 + 1-dimensional extreme black hole is zero, as was observed in the scalar case. We discuss how to construct the states at spatial infinity, where there is no plane wave solutions. The Dirac equation is solved for a special case where ω and n , the energy and azimuthal eigenvalues, respectively, satisfy a *fine tuning* condition.

The paper is organized as follows. In the following section we review the Dirac equation in a curved space time and a 2 + 1- dimensional extreme black hole. Section III is devoted to specify explicitly the Dirac equation in this background and to examine their asymptotic solutions. In Section IV the flux is constructed and the cross section is calculated; the reciprocal space approach is also discussed in this context. Finally, in Section V discussion and conclusions are presented.

II. THE DIRAC EQUATION IN A 2+1-DIMENSIONAL BLACK HOLE BACKGROUND

Let us consider a three dimensional Riemann manifold with Minkowski signature $(-, +, +)$ and line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (1)$$

The non-coordinate basis one-form $e^a = e^a_\mu dx^\mu$ and the affine spin connection $\omega_b^a = \omega_{b\mu}^a dx^\mu$ are defined by [14]

$$ds^2 = e^a e^b \eta_{ab}, \quad (2)$$

$$de^a + \omega_b^a e^b = 0, \quad (3)$$

where $\eta_{ab} = \text{diag}(-, +, +)$; latin indices denote tangent space components and greek indices stand for components of objects defined on the manifold.

The Dirac equation for a particle with mass m in the curved background (1) is given by [15]

$$\gamma^a E_a^\mu \left(\partial_\mu - \frac{1}{8} \omega_{bc\mu} [\gamma^b, \gamma^c] \right) \Psi = m \Psi, \quad (4)$$

where E_a^μ is the inverse triad which satisfies $E_a^\mu e_\mu^b = \delta_a^b$ and δ_a^b stands for the identity. $\{\gamma^a\}$ are the Dirac matrices in the tangent space defined by the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \quad (5)$$

A. Extreme 2 + 1 black hole metric

In order to compute the absorption cross section we must solve (4) in the three dimensional extreme black hole background.

In a 2 + 1-dimensional spacetime the Einstein equations with cosmological constant $\Lambda = -\ell^{-2}$ have a solution [8]

$$ds^2 = -N^2(r) dt^2 + N^{-2}(r) dr^2 + r^2 \left[d\phi + N^\phi(r) dt \right]^2, \quad (6)$$

where the lapse $N^2(r)$ and shift $N^\phi(r)$ functions are given by

$$N^2(r) = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}, \quad (7)$$

$$N^\phi(r) = -\frac{J}{2r^2}. \quad (8)$$

Here M and J are the mass and angular momentum of the black hole, respectively.

The lapse function vanishes when

$$r_{\pm} = r_{ex} \left[1 \pm \sqrt{1 - \frac{J^2}{M^2 \ell^2}} \right]^{\frac{1}{2}}, \quad (9)$$

and therefore, the solution (6) is defined for $r_+ < r < \infty$, $-\pi < \phi < \pi$ and $-\infty < t < \infty$.

The extreme solution corresponds to $J^2 = M^2 \ell^2$, in which case $r_{\pm} = r_{ex} = \ell \sqrt{M/2}$. Hence the line element can be written as

$$ds_{ex}^2 = -\left(\frac{r^2}{\ell^2} - 2 \frac{r_{ex}^2}{\ell^2} \right) dt^2 + \frac{\ell^2 r^2}{(r^2 - r_{ex}^2)^2} dr^2 - 2 \frac{r_{ex}^2}{\ell} dt d\phi + r^2 d\phi^2. \quad (10)$$

Instead of solving (4) with this metric, it is convenient to define a dimensionless set of coordinates $\{u, v, \rho\}$ as follows

$$u = \frac{t}{\ell} + \phi, \quad v = \frac{t}{\ell} - \phi, \quad e^{2\rho} = \frac{r^2 - r_{ex}^2}{\ell^2}, \quad (11)$$

where $-\infty < \{u, v\} < \infty$, $-\infty < \rho < \infty$. In the space $\{u \times v\}$, two points (u_1, v_1) and (u_2, v_2) are identified if they satisfy $u_1 = v_2$ and $v_1 = u_2$, for any value of ρ .

The line element (10) in these new coordinates reads

$$ds^2 = r_{ex}^2 dv^2 - \ell^2 e^{2\rho} du dv + \ell^2 d\rho^2, \quad (12)$$

and according to (2) we have the triads

$$e^1 = \frac{\ell^2 e^{2\rho}}{2 r_{ex}} du, \quad e^2 = r_{ex} dv - \frac{\ell^2 e^{2\rho}}{2 r_{ex}} du, \quad e^3 = \ell d\rho, \quad (13)$$

and from (3) the connections

$$\omega^1{}_2 = d\rho, \quad \omega^1{}_3 = \frac{\ell}{2} \frac{e^{2\rho}}{r_{ex}} du + \frac{r_{ex}}{\ell} dv, \quad \omega^2{}_3 = -\frac{\ell}{2} \frac{e^{2\rho}}{r_{ex}} du. \quad (14)$$

The non vanishing components of the inverse triad are

$$E_1^u = \frac{2}{\ell^2} r_{ex} e^{-2\rho}, \quad E_1^v = \frac{1}{r_{ex}} = E_2^v, \quad E_3^\rho = \frac{1}{\ell}. \quad (15)$$

In the next section we write the Dirac equation in these coordinates and will find its solution.

III. THE DIRAC EQUATION AND ITS SOLUTION

A. The Dirac Equation

We choose the Dirac matrices as follows

$$\gamma^1 = -i\sigma^3, \quad \gamma^2 = \sigma^1, \quad \gamma^3 = \sigma^2, \quad (16)$$

where σ^i are the Pauli matrices. This choice satisfies (5).

The Dirac equation (4) can be obtained directly using (13) through (15). If we write the solution as

$$\Psi(u, v, \rho) = \begin{pmatrix} \mathcal{U}(u, v, \rho) \\ \mathcal{V}(u, v, \rho) \end{pmatrix},$$

then (4) becomes

$$\left[-i \left(\frac{2r_{ex}e^{-2\rho}}{\ell^2} \partial_u + \frac{\partial_v}{r_{ex}} \right) - \left(\frac{1}{2\ell} + m \right) \right] \mathcal{U} + \left[\frac{\partial_v}{r_{ex}} - \frac{i}{\ell} (\partial_\rho - 1) \right] \mathcal{V} = 0, \quad (17)$$

$$\left[\frac{\partial_v}{r_{ex}} + \frac{i}{\ell} (\partial_\rho - 1) \right] \mathcal{U} + \left[i \left(\frac{2r_{ex}e^{-2\rho}}{\ell^2} \partial_u + \frac{\partial_v}{r_{ex}} \right) - \left(\frac{1}{2\ell} + m \right) \right] \mathcal{V} = 0. \quad (18)$$

In order to solve this equation, let us look for solutions of the form

$$\Psi(u, v, \rho) = e^{i(\alpha u + \beta v)} \begin{pmatrix} F(\rho) \\ G(\rho) \end{pmatrix}, \quad (19)$$

where α and β are constants related to the angular and temporal eigenvalues of the solution of (4) in the coordinates $\{t, \phi, r\}$; namely if the solution behaves like $e^{i(n\phi + \omega t)}$, then

$$\alpha = \frac{1}{2} (\omega\ell + n), \quad \beta = \frac{1}{2} (\omega\ell - n). \quad (20)$$

For the ρ -dependent part of the equation it is more convenient to define $z = e^{-2\rho}$. Therefore, using the Ansatz (19), the z -part of (17) and (18) becomes

$$\left[\frac{2\alpha r_{ex}}{\ell^2} z + \frac{\beta}{r_{ex}} - \frac{1}{2\ell} - m \right] F(z) + i \left[\frac{\beta}{r_{ex}} + \frac{1}{\ell} \left(2z \frac{d}{dz} + 1 \right) \right] G(z) = 0, \quad (21)$$

$$i \left[\frac{\beta}{r_{ex}} - \frac{1}{\ell} \left(2z \frac{d}{dz} + 1 \right) \right] F(z) - \left[2 \frac{\alpha r_{ex}}{\ell^2} z + \frac{\beta}{r_{ex}} + \frac{1}{2\ell} + m \right] G(z) = 0. \quad (22)$$

Here we have used the same notation for the functions F and G , independently whether they depend either on ρ or z . The notation will be used through out the text unless it becomes confusing.

One can find a solution of this set of first order coupled differential equations by rewriting them as a second order one. For example, if we solve (21) for $F(z)$ and replace this result in (22) we find that $G(z)$ satisfies a second order differential equation. By defining the variable $x = \alpha(r_{ex}/\ell) z$ one finds that $G(x)$ satisfies

$$A(x)G''(x) + B(x)G'(x) + C(x)G(x) = 0, \quad (23)$$

with

$$A(x) = (\delta - x)x^2, \quad (24)$$

$$B(x) = (2\delta - x)x, \quad (25)$$

$$C(x) = -x^3 + (\delta - \tilde{\beta})x^2 + \frac{1}{4}[(\tilde{\beta} + 1)^2 + 4\delta(2\tilde{\beta} + \delta)]x - \frac{\delta}{4}((2\delta + \tilde{\beta})^2 - 1), \quad (26)$$

where the constant δ , m_{eff} and $\tilde{\beta}$ are given by

$$\delta = \frac{1}{2}(\ell m_{\text{eff}} - \tilde{\beta}), \quad m_{\text{eff}} = m + \frac{1}{2\ell}, \quad \tilde{\beta} = \frac{\ell\beta}{r_{ex}}. \quad (27)$$

The same procedure yields a similar equation for $F(z)$.

Note that this equation has three singular points for $\delta \neq 0$. Two of them are regular (0 and δ) while the other one located at infinity is irregular (corresponding to the horizon in radial coordinates).

In order to solve the equation we must consider two cases: $\delta = 0$ and $\delta \neq 0$. In the first case, the equation has one regular singularity, while in the second one, it has the three singularities. Therefore, we will be able to completely solve the $\delta = 0$ case, while the other one must be treated in an approximate way.

The physical meaning of such cases is as follows. For s-waves, the $\delta = 0$ condition corresponds to particles with energies satisfying the relation $\omega = 2m_{\text{eff}}r_{\text{ex}}\Omega$, where $\Omega = 1/\ell$ is the angular velocity of the horizon. One can think of the previous condition as fixing the energy of the particle to the (classical) energy of a particle with mass m_{eff} rotating in a circular orbit of radius r_{ex} with angular velocity Ω . For waves with $n \neq 0$, the above relation holds, but now the energy has a contribution from the angular part.

Note that this condition yields a precise relation between the energy and the azimuthal eigenvalue and is a *fine tuning* condition that, for a generic wave, should be hard to fulfill.

In the next section we will discuss all cases in detail.

B. Solutions of Dirac equation

As we said previously, there exist two distinct cases. We will prove that, despite different solutions for both cases, they share a common characteristic related to their behaviour near the horizon.

1. $\delta = 0$.

In this case, equation (23) reads

$$x^2 G'' + xG' + \left[x^2 + \tilde{\beta}x - \frac{1}{4}(\tilde{\beta} + 1)^2 \right] G = 0, \quad (28)$$

with the following solution

$$G(x) = e^{-ix} x^{\frac{\tilde{\beta}+1}{2}} \left(P F \left[1 + \frac{1}{2}(1+i)\tilde{\beta}, 2 + \tilde{\beta}; 2ix \right] + Q U \left[1 + \frac{1}{2}(1+i)\tilde{\beta}, 2 + \tilde{\beta}; 2ix \right] \right), \quad (29)$$

where $F[c, d; x]$ and $U[c, d; x]$ are confluent hypergeometric functions (Kummer's solution); P and Q are complex constants. These functions provide a set of linearly independent solutions of (28) only if $\tilde{\beta}$ is non integer [16].

The confluent hypergeometric $F[c, d; x]$ is regular at $x = 0$, while the confluent hypergeometric $U[c, d; x]$ is regular at infinity. Thus, the regular piece of the solution at infinity (that is, near the horizon in radial coordinate) is

$$G(x) = e^{-ix} x^{\frac{\tilde{\beta}+1}{2}} Q U \left[1 + \frac{1}{2}(1+i)\tilde{\beta}, 2 + \tilde{\beta}; 2ix \right],$$

$$\sim e^{-ix} \sum_{n=0}^N \alpha_n x^{-n-1/2} \quad (x \rightarrow \infty), \quad (30)$$

where the α_n depend on $\tilde{\beta}$ and N is an arbitrary integer controlling the order of the expansion.

The regular solution for $x \rightarrow 0$ (that is, at spatial infinity) is $F[c, d; x]$ and has a series expansion in positive powers of x . However, the flux associated with this solution is not straightforwardly defined because the BTZ spacetime is not asymptotically flat; this is a subtle point because as was shown in [9]. We will discuss this issue in section IV.

2. $\delta \neq 0$.

In this case, the equation has three singular points, being $x \rightarrow \infty$ of irregular type. We can write (23) in a more transparent way by defining the variable $y = x/\delta$ and

$$G(y) = \frac{1-y}{\sqrt{y}} D(y). \quad (31)$$

The equation for $D(y)$ turns out to be

$$y^2(1-y)^2 D'' + y(1-y)(1-2y) D' + \left[\delta^2 y^4 + \delta (\tilde{\beta} - 2\delta) y^3 - \frac{\tilde{\beta}}{4} (2 + \tilde{\beta} + 12\delta) y^2 + \frac{1}{4} (5(\tilde{\beta} + \delta)^2 + 2\tilde{\beta}(1 + \delta) - 4) y - \frac{1}{4} (2\tilde{\beta} + \delta)^2 \right] D = 0. \quad (32)$$

This is the equation of a spheroidal wave function which has analytic solution for some particular cases only [17].

We are interested in the asymptotic behaviour of the solutions in order to define flux near the horizon and far from it. So, let us study the asymptotic form of (32).

We first analyse the $y \rightarrow 0$ case. Keeping terms up to first order, equation (32) can be written as

$$y^2 D_0'' + (1-y)y D_0' + \left(-\frac{1}{4} (2\tilde{\beta} + \delta)^2 + \frac{1}{4} (\tilde{\beta}(2 + 4\delta) - 3(\tilde{\beta}^2 - \delta^2 - 4)) y \right) D_0 = 0, \quad (33)$$

whose solution is

$$D_0(y) = P_0 y^{-\frac{b}{2}} F\left[\frac{1}{2}(a+b), 1-b, y\right] + Q_0 y^{\frac{b}{2}} F\left[\frac{1}{2}(a-b), 1+b, y\right], \quad (34)$$

where P_0 and Q_0 are complex constants, $a = 2 + \frac{3}{2}(\tilde{\beta}^2 - \delta^2) - \tilde{\beta}(1 + 2\delta)$ and $b = 2\tilde{\beta} + \delta$ and $F[c, d; x]$ is the confluent hypergeometric function.

Again, the states that define the incoming flux will be a superposition of this solutions.

In order to discuss the region $y \rightarrow \infty$, we make a series expansion of (32) about infinity and discard all terms of order $\mathcal{O}(y^{-1})$ and beyond. The resulting equation is

$$y^2 D_\infty'' + 2y D_\infty' + \left(\delta^2 y^2 + \delta \tilde{\beta} y - \frac{1}{4} (2\tilde{\beta} + \delta^2 + (\tilde{\beta} + \delta)(\tilde{\beta} + 3\delta)) \right) D_\infty = 0, \quad (35)$$

whose solutions turn out to be

$$D_\infty(y) = e^{-i\delta y} y^{\frac{1}{2}(a-1)} \left(P_\infty F \left[\frac{1}{2} (1 + i\tilde{\beta} + a), 1 + a, 2i\delta y \right] + Q_\infty U \left[\frac{1}{2} (1 + i\tilde{\beta} + a), 1 + a, 2i\delta y \right] \right), \quad (36)$$

where $a = \sqrt{(\tilde{\beta} + 1)^2 + 4\delta(\tilde{\beta} + \delta)}$. $F[c, d; y]$ and $U[c, d; y]$ are the confluent hypergeometric functions discussed previously for the case $\delta = 0$.

As we said in the previous section, the regular part of the solution is given by the $U[c, d; y]$ function. By the same arguments given there, we can write

$$D_\infty(y) \sim e^{-i\delta y} \sum_{n=0}^N \gamma_n y^{-n-1}, \quad (37)$$

where γ_n are complex constants. Therefore, the solution $G(x)$ turns out to be

$$G(x) \sim e^{-ix} \sum_{n=0}^N \tilde{\gamma}_n x^{-n-1/2}. \quad (38)$$

Finally, let us point out that the previous results can be understood from a physical point of view as follows. Because of our choice of radial coordinate, the near horizon region corresponds to the irregular singular point $x \rightarrow \infty$. But the horizon is not a physical singularity, and therefore one should find well behaved solutions near this point. Hence, the solution can be written as

$$G_\infty(x) \sim \frac{1}{x^s} \left(\tilde{\alpha}_0 + \frac{\tilde{\alpha}_1}{x} + \frac{\tilde{\alpha}_2}{x^2} + \dots \right), \quad (39)$$

where s is a positive real number. This is just the result found for $\delta = 0$ and $\delta \neq 0$ with $s = 1/2$. Note that the exponential e^{ix} cannot be obtained with this argument.

To close this section, let us comment on a question that may be confusing, that is, the fact that the Dirac equation can be *exactly* solved in the *regular* rotating BTZ background (see e.g. [18]). The difference with the case studied here is that the redefinition [11]

$$r^2 = r_+^2 \cosh^2 \mu - r_-^2 \sinh^2 \mu, \quad (40)$$

made to find the exact solution, has no meaning in the limit $r_- \rightarrow r_+$, that is, in the extreme BTZ background. This may be another expression that regular and extreme BTZ black holes correspond to different physical objects and should be studied separately.

IV. INGOING AND INCOMING FLUXES AND THE ABSORPTION RATE

As was mentioned before, it is not clear how to prepare an initial scattering state (incoming flux) in an AdS background because the solution of the equations of motion at infinity ($r \rightarrow \infty$) are not plane waves. If one needs to find a consistent expression for the incoming flux at infinity, one could define the incoming flux by following the procedure developed in [9] for a scalar field but this is of no help here, as we will show below. The problem can be circumvented arguing that the incoming flux, must be non zero because it represents particles that are sent to the black hole. Therefore, we only must care to be far enough from the horizon. We will use here the arguments given in [11] for the case of fermions scattering off a regular BH: since in an asymptotically flat space $\sqrt{g_{oo}} \rightarrow 1$ at infinity, and since $\sqrt{g_{oo}} \rightarrow r$ for a BTZ background, one chooses a point in space a location such that $\sqrt{g_{oo}} \rightarrow 1$; this occurs for $r \sim l$.

A. The current

In order to calculate the absorption cross section, it is necessary to prepare the incoming flux (initial state) and to obtain the ingoing flux (near r_{ex}) later. The current along ρ -direction is

$$j^\rho = \kappa \bar{\psi} \gamma^a E_a^\rho \psi, \quad (41)$$

$$= -\frac{2\kappa}{\ell} \Re(\mathcal{U}^* \mathcal{V}), \quad (42)$$

where κ is a coupling constant.

From (19) and (21) we obtain

$$j^\rho(x) = \mathcal{A} \left(\frac{x}{\delta - x} \right) \Re \left\{ i G(x) \frac{d}{dx} G^*(x) \right\}, \quad (43)$$

where \mathcal{A} is a constant.

B. Flux and cross section

The black hole absorption rate σ_{abs} could be defined as in quantum mechanics, where it is related to the ratio of the ingoing flux at the horizon (total number of particles entering the horizon) and incoming flux at infinity.

$$\sigma_{abs} = \frac{\mathcal{F}(x \rightarrow \infty)}{\mathcal{F}(x \rightarrow 0)}, \quad (44)$$

where

$$\mathcal{F} = \sqrt{-g} j^\rho(x), \quad (45)$$

with g the determinant of the metric.

It is straightforward to prove that

$$\mathcal{F} = \mathcal{A} \sqrt{r_{ex}^2 + \frac{\alpha \ell r_{ex}}{x}} \left(\frac{x}{1-x} \right) \Re \left\{ i G(x) \frac{d}{dx} G^*(x) \right\}. \quad (46)$$

Now we must evaluate this quantity for $\delta = 0$ and $\delta \neq 0$, near the horizon and at spatial infinity. As we showed in the previous section, however, there is a general form of the solution given by (39), as we showed in the previous section for the near horizon region.

It is straightforward to evaluate the flux near the horizon:

$$\begin{aligned} \mathcal{F} &= \mathcal{A} \sqrt{r_{ex}^2 + \frac{\alpha \ell r_{ex}}{x}} \frac{x}{1-x} \Re \left\{ \frac{i}{x^{2s}} \left(\tilde{\alpha}_0 + \frac{\tilde{\alpha}_1}{x} + \frac{\tilde{\alpha}_2}{x^2} + \dots \right) \left(\kappa_0 + \frac{\kappa_1}{x} + \frac{\kappa_2}{x^2} \dots \right) \right\} \\ &= \mathcal{A} \sqrt{r_{ex}^2 + \frac{\ell^2}{x}} \frac{1}{1-x} \Re \left\{ \frac{i}{x^{2s-1}} \left(\tilde{\alpha}_0 \kappa_0 + \frac{\tilde{\alpha}_0 \kappa_1 + \tilde{\alpha}_1 \kappa_0}{x} + \frac{\tilde{\alpha}_0 \kappa_2 + \tilde{\alpha}_1 \kappa_1 + \tilde{\alpha}_2 \kappa_0}{x^2} \dots \right) \right\}. \end{aligned}$$

Here $\tilde{\alpha}_i$ and $\kappa_0 = i\tilde{\alpha}_0^*$, $\kappa_1 = i\tilde{\alpha}_1^* - s\tilde{\alpha}_0^*$, $\kappa_2 = i\tilde{\alpha}_2^* - \tilde{\alpha}_1^* - s\tilde{\alpha}_1^*$, \dots , are numerical constants.

The last expression shows that for $s \geq 1/2$

$$\lim_{x \rightarrow \infty} \mathcal{F} \rightarrow 0. \quad (47)$$

In order to define the incoming flux in a consistent way, one could define the ingoing and outgoing states as a complex linear combination of the asymptotic solutions at infinity, as done in [9] for a scalar field. In the present case, ($\delta = 0$) this means to consider

$$G_{r \rightarrow \infty}(x) = e^{-ix} x^{\frac{\tilde{\beta}+1}{2}} (PF[1 + \frac{1}{2}(1+i)\tilde{\beta}, 2 + \tilde{\beta}; 2ix] \pm QU[1 + \frac{1}{2}(1+i)\tilde{\beta}, 2 + \tilde{\beta}; 2ix]), \quad (48)$$

where P y Q are complex constants and the positive (negative) sign corresponds to the ingoing (outgoing) waves. It is straightforward to show that, by replacing this expression

into Eq. (46), the flux diverges (to leading order) as $x^{-\frac{1}{2}}$ at infinity, a result that is valid for $\delta = 0$ as well as $\delta \neq 0$.

Given that the previously defined flux is divergent, we can use the argument found in [11] in order to define the incoming flux. Choosing $x \sim \text{const. } l$ (corresponding to $r \sim l$) we find

$$\lim_{r \rightarrow l} \mathcal{F} \rightarrow \text{constant}. \quad (49)$$

Finally, from (44) one proves that

$$\sigma_{abs} = 0. \quad (50)$$

for the extreme 2+1 dimensional black hole.

Let us point out that this result has no contradictions with the non-extreme black hole. The reason is that our argument about the form of the function $G(x)$ is still valid in the regular black hole background, but the damping factor in front of (43) is different for such case because Eqs. (21) and (22) are not the same for the extreme and non-extreme cases and hence one cannot map one into the other in a continuous manner.

C. Reciprocal space

In [1] we argued that it is possible to circumvent the problem of defining the flux of particles at infinity in a AdS spacetime by working in a sort of reciprocal space, which means to map the original problem into another one, where the concept of free particles at infinity holds.

In the present case this can also be performed. The arguments given before –in [1]– are still valid and much of the discussion in this section is based on it.

The main issue is to write the equation (23) in its canonical form

$$u''(x) + I(x)u(x) = 0, \quad (51)$$

where $I(x)$ is the invariant.

Again, in our problem we must distinguish two cases: $\delta = 0$ and $\delta \neq 0$.

For $\delta = 0$, one defines

$$G(x) = \frac{u_0(x)}{\sqrt{x}}, \quad (52)$$

where $u_0(x)$ satisfies the equation

$$u_0'' + \left(1 + \frac{\tilde{\beta}}{x} - \frac{\tilde{\beta}(\tilde{\beta} + 2)}{4x^2}\right) u_0 = 0. \quad (53)$$

For $\delta \neq 0$, defining $\xi = x/\delta$ and

$$G(\xi) = \frac{\sqrt{\xi - 1}}{\xi} u_1(\xi), \quad (54)$$

one finds that $u_1(\xi)$ satisfies

$$u_1'' + \left(\delta^2 + \frac{\tilde{\beta}(2\delta + 1) - 1}{2\xi} + \frac{(\tilde{\beta} + 2\delta)^2 - 1}{4\xi^2} + \frac{1 - \tilde{\beta}}{1 - \xi} + \frac{(3/4)}{(\xi - 1)^2}\right) u_1 = 0. \quad (55)$$

In (53) and (55), x and ξ play the role of radial coordinates, respectively, in a Schrödinger-type equation. The reciprocal spaces are then $\mathcal{H}_0 = \{\phi, x\}$ and $\mathcal{H}_1 = \{\phi, \xi\}$, where ϕ is an angular coordinate.

Additionally, one must impose that

$$u_0(x = 0) = 0 = u_1(\xi = 0),$$

in order to have continuous solutions everywhere [19].

Finally, one note that the *potential* terms in (53) and (55) vanish as $x \rightarrow \infty$ and $\xi \rightarrow \infty$, respectively. Therefore, it is possible to define asymptotic states as in the usual scattering theory; meaning solutions like $A_0(\phi)e^{ix}$ for u_0 and $A_1e^{i\delta\xi}$ for u_1 , where A_i are the scattering amplitudes and the exponential terms, are the asymptotic states.

However, invoking the optical theorem, one can directly prove that this gives rise to a vanishing total cross section (see [1] for details). This implies a zero absorption cross section as was calculated in the previous section when returning to the original space

V. CONCLUSIONS

In this work we have discussed the problem of fermions scattering off a 2+1-dimensional extreme black hole background.

We showed that the Dirac equation can be solved when the *fine tuning condition* is satisfied.

We also gave a general argument in order to show that the solutions of the Dirac equation near the horizon need to have a precise form, what was checked by solving them.

With these solutions we were able to prove that the flux of incoming particles near the horizon vanishes and therefore that the absorption cross section becomes zero, which agrees with previous results for scalar particles.

The previous result is not contradictory with the well known fact that for a regular BH background, the cross section is proportional to the area (entropy) of the horizon. In fact, our result is consistent with the fact that both solutions of the Einstein equations (the regular and the extreme black holes) define different topologies of the spacetime.

We also proved that the description in terms of the *reciprocal space* yields the same results implying that it could be considered as the space where the scattering in AdS should be defined.

Finally, let us remark that our main result agrees with arguments that show that the extreme black hole entropy is zero and therefore can be considered as fundamental objects.

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